Fibonacci Numbers and Binet's Formula using Linear Algebra

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1 Fibonacci Numbers

A *sequence* is a list that never ends (e.g. $\{1, 3, 5, 7, 9, 13, \ldots\}$) The *Fibonacci Sequence* is an exciting sequence discovered almost 1000 years ago. The first few numbers are:

f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	$\ldots$
0	1	1	2	3	5	8	13	21	34	$\ldots$

The rule that makes the Fibonacci Sequence is "the next number is the sum of the previous two". This kind of rule is sometimes called a *recurrence relation.* Mathematically, this is written as:

$$f_n = f_{n-1} + f_{n-2}$$

There is an explicit formula for the n-th Fibonacci number known as **Binet's formula:**

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

In the rest of this note, we will use **linear algebra** to derive **Binet's formula** for the Fibonacci numbers. This will partial explain where these mysterious numbers in the formula come from. The main tool is to rewrite the Fibonacci numbers using **matrix multiplication** and use some tools from linear algebra for simplifying matrix multiplication.

2 Converting the Fibonacci numbers to linear algebra

To use the power of linear algebra on the Fibonacci numbers, we define a sequence of **two-dimensional Fibonacci vectors** $\vec{v}_1, \vec{v}_2 \ldots$ that contain the Fibonacci sequence, which we define as follows:

$$\vec{v}_n = \begin{bmatrix} f_n \\ f_{n+1} \end{bmatrix}$$

The first few vectors are in this list (compare this to the list of Fibonacci numbers to see the pattern):

$\vec{v}_0$	$\vec{v}_1$	$\vec{v}_2$	$\vec{v}_3$	$\vec{v}_4$	$\vec{v}_5$	$\vec{v}_6$	$\vec{v}_7$	$\vec{v}_8$	$\ldots$
$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 5 \end{bmatrix}$	$\begin{bmatrix} 5 \\ 8 \end{bmatrix}$	$\begin{bmatrix} 8 \\ 13 \end{bmatrix}$	$\begin{bmatrix} 13 \\ 21 \end{bmatrix}$	$\begin{bmatrix} 21 \\ 34 \end{bmatrix}$	$\ldots$

Because of the definition of $\vec{v}_n$ and the update rule $f_n = f_{n-1} + f_{n-2}$ for the Fibonacci numbers, the rule for going from one vector $\vec{v}_n$, to the next vector $\vec{v}_{n+1}$ is given by the following two rules:

- The *first component* of $\vec{v}_{n+1}$ is equal to the *second component* of $\vec{v}_n$
- The *second component of* $\vec{v}_{n+1}$ is equal to the *sum of the first and second component* of $\vec{v}_n$.

These two rules can be written down all at once as **matrix multiplication**! The update rule is equivalent to:

$$\vec{v}_{n+1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \vec{v}_n$$

We can use this update rule a bunch of times to write everything in terms of **powers of** the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and the starting point $\vec{v}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. For example for $\vec{v}_3$ we have

$$\begin{aligned}
\vec{v}_3 &= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \vec{v}_2 \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \vec{v}_1 \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \vec{v}_0 \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^3 \vec{v}_0
\end{aligned}$$

In general, the formula for $\vec{v}_n$ is just:

$$\vec{v}_n = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We are now really close to Binet's formula! We just need to find an explicit formula for the matrix power $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n$.

3 Explicit formula for a matrix power

In this section we will find an explicit for $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n$. The key to finding the explicit formula for a matrix power is to use a special matrix decomposition. It is a fact that the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ can be written as a product of three matrices as follows:

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \frac{-1+\sqrt{5}}{2\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{-1-\sqrt{5}}{2\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{bmatrix}$$

Why would anyone write such a nasty looking product?!?! In fact this decomposition is very nice because the **first** and **last** matrix in this product are **inverses** of each other:

$$\begin{bmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \frac{-1+\sqrt{5}}{2\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1+\sqrt{5}}{2\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{-1+\sqrt{5}}{2\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1+\sqrt{5}}{2\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For this reason it is more visually convenient to call this matrix "B" and just write:

$$\begin{bmatrix} \frac{-1+\sqrt{5}}{2\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1+\sqrt{5}}{2\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{bmatrix} = B^{-1}$$

$$\begin{bmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{bmatrix} = B$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = B \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} B^{-1}$$

This inverse property of the first and last matrix means that when you write out the matrix power you get a remarkable simplification. For example:

$$
\begin{aligned}
\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^2 &= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \\
&= \left(B\begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}B^{-1}\right)\left(B\begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}B^{-1}\right) \\
&= B\begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}(B^{-1}B)\begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}B^{-1} \\
&= B\begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}B^{-1} \\
&= B\begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}\begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}B^{-1} \\
&= B\begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^2 & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^2 \end{bmatrix}B^{-1}
\end{aligned}
$$

The last step works out easily because the middle matrix in our decomposition is a **diagonal** matrix. Multiplying diagonal matrices is very easy! This type of decomposition we used, $M = BDB^{-1}$ where B and B^{-1} are inverses of each other and D is a diagonal matrix, is called **diagonalizing a matrix**. The entries of the diagonal matrix D are the **eigenvalues** of the matrix. The columns of B are the **eigenvectors**. Some more details about how to calculate the matrix D and the matrix B are given in the next section.

As we've seen here for the Fibonacci numbers, this trick makes it possible to compute the powers of the matrix. The same trick works for **any power**

$$
\begin{aligned}
&\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n \\
=&\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\dots\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \\
=&\left(B\begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}B^{-1}\right)\left(B\begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}B^{-1}\right)\dots\left(B\begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}B^{-1}\right) \\
=&B\begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}(B^{-1}B)\dots(B^{-1}B)\begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}B^{-1} \\
=&B\begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\dots\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}B^{-1} \\
=&B\begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{bmatrix}B^{-1}
\end{aligned}
$$

So finally we get the explicit formula for the Fibonacci vectors $\vec{v}_n$ as

$$
\begin{aligned}
\vec{v}_n &= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= B \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{bmatrix} B^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{bmatrix} \begin{bmatrix} \frac{-1+\sqrt{5}}{2\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{-1-\sqrt{5}}{2\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n \\ \frac{-1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n \end{bmatrix}
\end{aligned}
$$

Finally we can read off the explicit formula for the n-th Fibonacci number f_n because the first component of $\vec{v}_n$ is exactly f_n:

$$
\begin{aligned}
f_n &= \text{first component of } \vec{v}_n \\
&= \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n
\end{aligned}
$$

This is Binet's formula!!!

4 How to find the decomposition of the matrix

In this section I'll show some of the calculations that are needed to get the decomposition which we used to find Binet's formula:

$$
\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \frac{-1+\sqrt{5}}{2\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{-1-\sqrt{5}}{2\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{bmatrix}
$$

4.1 Disclaimer on Eigenvalues and Eigenvectors

Eigenvalues and eigenvectors are a fundemental tool for dealing with matrices! In this section, I'm going to show the calculation you would do to get the matrix we used. However, if you've never seen eigenvalues/eigenvectors before it will probably not make much sense. You might be interested in watching the 3Blue1Brown video on eigenvalues/eigenvectors `https://www.youtube.com/watch?v=PFDu9oVAE-g` on eigenvectors and eigenvalues to get a more intuitive feel for this before jumping into these calculations.

4.2 Characteristic polynomial and eigenvalues

We start by computing the **characteristic polynomial** of the matrix:

$$
\begin{aligned}
p(\lambda) &= \det\left(\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \\
&= \det\left(\begin{bmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} \right) \\
&= -\lambda\left(1-\lambda\right) - 1 \\
&= \lambda^2 - \lambda - 1
\end{aligned}
$$

We then find the **eigenvalues** as the roots of the characteristic polynomial using the quadratic formula:

$$
\lambda_1 = \frac{1+\sqrt{5}}{2}, \lambda_2 = \frac{1-\sqrt{5}}{2}
$$

We then solve for the eigenvalues by find the nullspace of $M - \lambda I$

4.3 Nullspace associated to λ_1

We solve for the vector $\vec{x}$ in

$$\left(\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} - \left(\frac{1+\sqrt{5}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\iff \left(\frac{1+\sqrt{5}}{2}\right)x_1 = x_2$$

So $x_1 = 1$, $x_2 = \frac{1+\sqrt{5}}{2}$ is a valid eigenvector.

$$\vec{e}_1 = \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix}$$

4.4 Nullspace associated to λ_2

We solve for the vector $\vec{x}$ in

$$\left(\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} - \left(\frac{1-\sqrt{5}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\iff \left(\frac{1-\sqrt{5}}{2}\right)x_1 = x_2$$

So $x_1 = 1$, $x_2 = \frac{1+\sqrt{5}}{2}$ is a valid eigenvector.

$$\vec{e}_2 = \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}$$

4.5 Putting it together

In the basis $\vec{e}_1$,$\vec{e}_2$ we have that the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ is diagonal with entries λ_1, λ_2. By the **change of basis** result for writing a matrix in the basis $\vec{e}_1$,$\vec{e}_2$, we have then:

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{bmatrix}\begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{bmatrix}\begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}\begin{bmatrix} \frac{-1+\sqrt{5}}{2\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1+\sqrt{5}}{2\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{bmatrix}$$

In the last step, I've used the formula for a 2×2 inverse matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$
$$\implies \begin{bmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{bmatrix}^{-1} = \frac{1}{-\sqrt{5}}\begin{bmatrix} \frac{1-\sqrt{5}}{2} & -1 \\ -\frac{1+\sqrt{5}}{2} & 1 \end{bmatrix}$$