A polynomial equation may have a repeated root. Here are two examples:

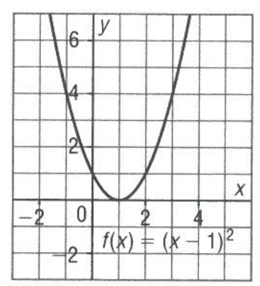
$x^2 - 2x + 1 = 0$ can be written as $(x - 1)^2 = 0$.
The equation has root: $x = 1$
The exponent of the factor $(x - 1)$ is 2, so 1 is a root with **multiplicity** 2.
The related function has a zero of multiplicity 2.

$x^3 - 3x^2 + 3x - 1 = 0$ can be written as $(x - 1)^3 = 0$.
The equation has root: $x = 1$
The exponent of the factor $(x - 1)$ is 3, so 1 is a root with multiplicity 3.
The related function has a zero of multiplicity 3.

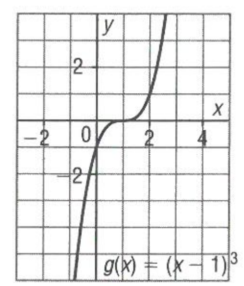
The behaviour of the graph at a zero depends on its multiplicity.

Here are the graphs of $f(x) = (x - 1)^2$ and $g(x) = (x - 1)^3$.

This graph has a zero of multiplicity 2.



This graph has a zero of multiplicity 3.



Both graphs have x-intercept 1.
The graph of $f(x) = (x - 1)^2$ touches the x-axis at $x = 1$, but does not cross the axis at this point.
The graph of $g(x) = (x - 1)^3$ crosses the x-axis at $x = 1$.
This difference in behaviour is related to the multiplicity of the zero.
In general, when a zero has even multiplicity, the graph touches the x-axis at the related x-intercept, but does not cross it; we say that the graph "just touches" the x-axis.
When a zero has odd multiplicity, the graph crosses the x-axis at the related x-intercept.

TEACHER NOTE

DI: Extending Thinking

Use the graph of $y = (x - 1)^3$ to introduce the concept of a point of inflection. If you have previously introduced the terms concave up and concave down, you could define the point of inflection as the point where a graph changes from concave up to concave down, or vice versa.

Example 3 Using the Multiplicity of a Zero to Graph a Polynomial Function

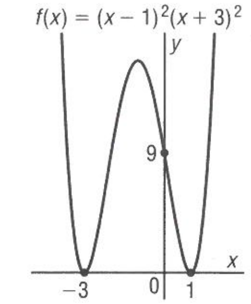
Animation

Sketch the graph of each polynomial function.

a) $f(x) = (x - 1)^2(x + 3)^2$ **b)** $g(x) = -(x + 2)^3(x - 1)^2$

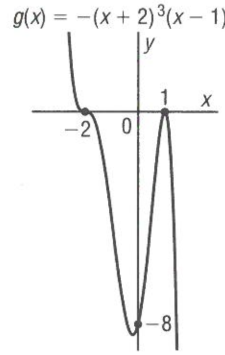
SOLUTION

a) $f(x) = (x - 1)^2(x + 3)^2$



To determine the zeros, solve $f(x) = 0$.
$0 = (x - 1)^2(x + 3)^2$
The roots of the equation are $x = 1$ and $x = -3$.
So, the zeros of the function are 1 and -3.
The zero 1 has multiplicity 2.
The zero -3 has multiplicity 2.
So, the graph just touches the x-axis at $x = 1$ and at $x = -3$.
The equation has degree 4, so it is an even-degree polynomial function. The leading coefficient is positive, so the graph opens up.
The y-intercept is: $(-1)^2(3)^2 = 9$
Plot points at the intercepts, then draw a smooth curve that rises to the left and rises to the right.

b) $g(x) = -(x + 2)^3(x - 1)^2$



To determine the zeros, solve $g(x) = 0$.
$0 = -(x + 2)^3(x - 1)^2$
The roots of the equation are $x = -2$ and $x = 1$.
So, the zeros of the function are -2 and 1.
The zero -2 has multiplicity 3.
The zero 1 has multiplicity 2.
So, the graph crosses the x-axis at $x = -2$, and just touches the x-axis at $x = 1$.
The equation has degree 5, so it is an odd-degree polynomial function.
The leading coefficient is negative, so as $x \to -\infty$, the graph rises and as $x \to \infty$, the graph falls.
The y-intercept is: $-(2)^3(-1)^2 = -8$
Plot points at the intercepts, then draw a smooth curve that rises to the left and falls to the right.

Check Your Understanding

3. Sketch the graph of each polynomial function.

a) $f(x) = (x + 1)^4(x - 2)$

b) $g(x) = -(x + 1)^3(x - 3)$

a) Let $f(x) = 0$.
$0 = (x + 1)^4(x - 2)$
The zeros of the function are -1 and 2. The zero -1 has multiplicity 4.
The zero 2 has multiplicity 1.
So, the graph just touches the x-axis at $x = -1$ and crosses the x-axis at $x = 2$. The equation has degree 5, so it is an odd-degree polynomial function. The leading coefficient is positive, so as $x \to -\infty$, the graph falls and as $x \to \infty$, the graph rises. The y-intercept is: $(1)^4(-2) = -2$
Draw a smooth curve that falls to the left and rises to the right.

b) Let $g(x) = 0$.
$0 = -(x + 1)^3(x - 3)$
The zeros of the function are -1 and 3. The zero -1 has multiplicity 3. The zero 3 has multiplicity 1. So, the graph crosses the x-axis at $x = -1$ and $x = 3$.
The equation has degree 4, so it is an even-degree polynomial function. The leading coefficient is negative, so the graph opens down. The y-intercept is: $-(1)^3(-3) = 3$
Draw a smooth curve that falls to the left and falls to the right.